"The three points in which any line cuts the sides of a triangle, and the projections, from any point in the plane, of the vertices of the triangle on the same line are six points in involution."

The fact that this proposition has thus been derived from another as its reciprocal, will not at all interfere with the deduction in the ordinary manner of the reciprocal theorem concerning six rays in involution joining any point to the vertices of a complete quadrilateral.

On the other hand, as no use whatever has been made of harmonic properties to obtain the theorem, these may all be deduced as special applications, by drawing the transversal AX through two of the points in which sides of the triangle MNP meet the lines joining Q to the opposite vertices, thus making these points foci of the involution.

Or, should it be preferred to proceed more nearly in the usual course, a definition of the punctual distance, equivalent to the one already stated, may be given in terms of angular functions. Thus, in the figure, the punctual distance between NC' and NB is

$$\frac{\sin NAX \sin BNC'}{N'N \sin ANB \sin ANC'};$$

and it is easily shown that the ratio in which the punctual distance of two out of three convergents is divided by the third is equal to the anharmonic ratio of the pencil formed by adding to the three convergents a line which joins the point of convergence to the origin.

A NEW AND USEFUL FORMULA FOR INTEGRATING CERTAIN DIFFERENTIALS.

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PROBLEM.—*To integrate* $v^n du$ *in terms of the descending powers of* v.

Let us assume

$$\int v^n du = yv^n + y_1 v^{n-1} + y_2 v^{n-2} + \ldots + y_t v^{n-t} + \int z v^{n-t-1} dv. \qquad (1)$$

Differentiating (1), and arranging with reference to v, we have

$$0 = \left.\begin{matrix} -du \\ dy \end{matrix}\right| v^n + \left.\begin{matrix} nydv \\ dy_1 \end{matrix}\right| v^{n-1} + \left.\begin{matrix} (n-1)y_1 dv \\ dy_2 \end{matrix}\right| v^{n-2} + \ldots + \left.\begin{matrix} (n-t)y_t dv \\ zdv \end{matrix}\right| v^{n-t-1} \qquad (2)$$

Now since (2) is true for every value of v, according to the principle of indeterminate coefficients, we have

$$\left.\begin{aligned} dy &= du, \\ dy_1 &= -nydv, \\ dy_2 &= -(n-1)y_1dv, \\ dy_3 &= -(n-2)y_2dv, \\ &\ldots\ldots\ldots\ldots \\ dy_n &= -(n-t+1)y_{t-1}dv, \\ z &= -(n-t)y_t. \end{aligned}\right\} \quad (3)$$

Integrating equations (3), supposing the constant to be added $= 0$, and denoting

$$\begin{aligned} &\int dv \int udv \quad \text{by } \int^2 udv^2, \\ &\int dv \int dv \int udv \text{ by } \int^3 udv^3, \\ &\text{\&c.} \qquad \text{\&c.} \qquad \text{\&c.,} \end{aligned}$$

we have

$$\left.\begin{aligned} y &= u \\ y_1 &= (-1)^1 n \int udv \\ y_2 &= (-1)^2 n(n-1)\int^2 udv^2 \\ y_3 &= (-1)^3 n(n-1)(n-2)\int^3 udv^3 \\ &- - - - - - - - - - - - - - \\ y_t &= (-1)^t n(n-1)\ldots(n-t+1)\int^t udv^t \\ z &= (-1)^{t+1} n(n-1)\ldots(n-t)\int^t udv_t. \end{aligned}\right\} \quad (4)$$

Substituting in (1), we have

$$\int v^n du = uv^n - nv^{n-1}\int udv + n(n-1)v^{n-2}\int^2 udv^2 - n(n-1)(n-2)v^{n-3}\int^3 udv^3$$
$$+ \ldots (-1)^t n(n-1)\ldots(n-t+1)v^{n-t}\int^t udv^t + \ldots (-1)^{t+1} n(n-1)\ldots$$
$$(n-t)\int v^{n-t-1} dv \int^t udv^t. \quad (5)$$

If $t = n$ (5) becomes

$$\int v^n du = uv^n - nv^{n-1}\int udv + n(n-1)v^{n-2}\int^2 udv^2 \ldots (-1)^n n(n-1)\ldots$$
$$\ldots(3)(2)(1)\int^n udv^n. \quad (6)$$

Of course (6) is not so useful and general as (5), but the want of space will scarcely allow us to indicate the utility of the former in the applications of the formula which we purpose make to the integration of a few important differentials.

Ex. 1. $\int x^m \log^n x\, dx = ?$

Make $u = x^{m-1} \div (m+1)$, and $v = \log x$. Hence,

$$udv = \frac{x^m dx}{m+1}, \quad \int udv = \frac{x^{m+1}}{(m+1)^2};$$

$$dv\int udv = \frac{x^m dx}{(m+1)^2}, \quad \int^2 udv^2 = \frac{x^{m+1}}{(m+1)^3}.$$

Similarly,

$$\int^3 udv^3 = \frac{x^{m+1}}{(m+1)^4},$$
$$\ldots\ldots\ldots$$
$$\int^n udv^n = \frac{x^{m+1}}{(m+1)^n}.$$

Substituting in (6), we have

$$\int x^m \log^n x dx = \frac{x^{m+1}}{m+1}\left[\log^n x - \frac{n\log^{n-1}x}{m+1} + \frac{n(n-1)\log^{n-2}x}{(m+1)^2} \cdots \right.$$

$$\left. (-1)^n \frac{n(n-1)\ldots(3)(2)(1)}{(m+1)^n}\right]. \quad (7)$$

If in (7) $m=0$, and we take the integral between the limits $x=0$ and $x=1$, we have

$$\int_0^1 \log^n x dx = (-1)^n n(n-1)(n-2)\ldots 3 \,.\, 2 \,.\, 1.$$

Ex. 2. $\int a^x x^n dx = ?$

Make $u = a^x \div (\log a)$, and $v = x$. Hence,

$$\int udv = \frac{a^x}{\log^2 a},$$

$$\int^2 udv^2 = \frac{a^x}{\log^3 a},$$

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$$\int^n udv^n = \frac{a^x}{\log^{n+1} a}.$$

Substituting in (6),

$$\int a^x x^n dx = \frac{a^x}{\log a}\left[x^n - \frac{nx^{n-1}}{\log a} + \frac{n(n-1)x^{n-2}}{\log^2 a} \ldots (-1)^n \frac{n(n-1)\ldots 3.2.1}{\log^n a}\right].$$

In a similar manner we may integrate a number of other important differentials, such as

$$\sin^m x \cos^n x dx, \qquad (a+bx^n)^p x^m dx, \ \&c.$$

Solution of Prob. 334, by W. E Heal.—(See pp. 60, 98, Vol. VIII)

Let the given ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

The locus of intersection of tangents which intersect at a constant angle, φ, is

$$[(x^2+y^2)-(a^2+b^2]^2\tan^2\varphi = 4(b^2x^2+a^2y^2-a^2b^2). \qquad (A)$$

(Salmon's Conics, 5th edition, page 161.)

The polar of every point on this curve touches the required curve and conversely the polar of every point on the required curve touches the curve (A).

Let ($\varkappa$, λ) be the coordinates of any point on the required curve.

The polar of this point is

$$\frac{\varkappa x}{a^2} + \frac{\lambda y}{b^2} = 1. \qquad (B)$$